A Hot Pot

Lin Jiu

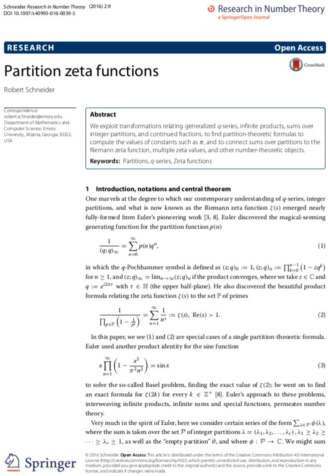
RISC

Algorithmic Combinatorics Seminar

Oktober 5^{th} 2016

Outlines

Beginning-Partition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Partition zeta functions

Robert Schneider

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Abstract

We exploit transformations relating generalized q-series, infinite products, sums over integer partitions, and continued fractions, to find partition-theoretic formulas to compute the values of constants such as π, and to connect sums over partitions to the Riemann zeta function, multiple zeta values, and other number-theoretic objects.

Keywords: Partitions, q-series, Zeta functions

1 Introduction, notations and central theorem

One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function $\zeta(s)$ emerged nearly fully-formed from Euler's pioneering work [3, 8]. Euler discovered the magical-seeming generating function for the partition function $p(n)$

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n, \tag{1}$$

in which the q-Pochhammer symbol is defined as $(z;q)_0 := 1$, $(z;q)_n := \prod_{k=0}^{n-1}(1-zq^k)$ for $n \geq 1$, and $(z;q)_\infty = \lim_{n\to\infty}(z;q)_n$ if the product converges, where we take $z \in \mathbb{C}$ and $q := e^{i2\pi\tau}$ with $\tau \in \mathbb{H}$ (the upper half-plane). He also discovered the beautiful product formula relating the zeta function $\zeta(s)$ to the set $\mathbb{P}$ of primes

$$\frac{1}{\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)} = \sum_{n=1}^{\infty}\frac{1}{n^s} := \zeta(s), \quad \mathrm{Re}(s) > 1. \tag{2}$$

In this paper, we see (1) and (2) are special cases of a single partition-theoretic formula. Euler used another product identity for the sine function

$$x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = \sin x \tag{3}$$

to solve the so-called Basel problem, finding the exact value of $\zeta(2)$; he went on to find an exact formula for $\zeta(2k)$ for every $k \in \mathbb{Z}^+$ [8]. Euler's approach to these problems, interweaving infinite products, infinite sums and special functions, permeates number theory.

Very much in the spirit of Euler, here we consider certain series of the form $\sum_{\lambda\in\mathcal{P}}\phi(\lambda)$, where the sum is taken over the set $\mathcal{P}$ of integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, as well as the "empty partition" $\emptyset$, and where $\phi : \mathcal{P} \to \mathbb{C}$. We might sum

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- Special cases:

1. $$\frac{1}{(q;q)_\infty} = \sum_{n\geq 0} p(n) q^n.$$

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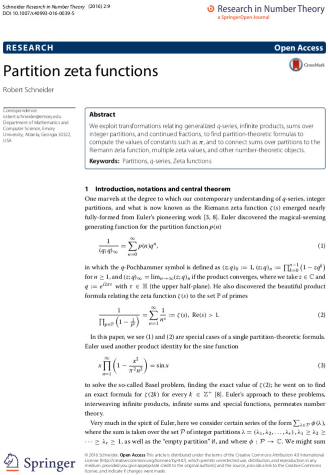

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Partition zeta functions

DEF: partition-theoretic zeta function

For $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we denote

$$l(\lambda) := k \text{ and } n_\lambda := \lambda_1 \cdots \lambda_k$$

Define the *partition-theoretic generaliztion of Riemann-zeta function* as

$$\zeta_{\mathcal{P}}\left(\{s\}^k\right) := \sum_{l(\lambda)=k} \frac{1}{n_\lambda^s}.$$

Theorem

$$\zeta_{\mathcal{P}}\left(\{2\}^k\right) = \sum_{l(\lambda)=k} \frac{1}{n_\lambda^2} = \frac{2^{2k-1}-1}{2^{2k-2}} \zeta(2k).$$

- Similar results for $\zeta_{\mathcal{P}}\left(\{2m\}^k\right) = \text{rational} \times \pi^{2k}$.
- The author wants to study general $\zeta_{\mathcal{P}}\left(\{s\}^k\right)$ and its AC.

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$$S_{a_1,\dots,a_k}(N) = \sum_{N \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{\operatorname{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \cdots \frac{\operatorname{sign}(a_k)^{i_k}}{i_k^{|a_k|}}.$$

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Let $f(n)=\frac{t^a}{n^a}$ and $q\to 1$

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In particular, if $a=m\in\mathbb{N}$ and $m\geq 2$, by considering $\xi_m:=\exp\left(\frac{2\pi\iota}{m}\right)$

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$$\alpha_1+\cdots+\alpha_m=\beta_1+\cdots+\beta_m\Rightarrow\prod_{k\geq 0}\frac{(k+\alpha_1)\cdots(k+\alpha_m)}{(k+\beta_1)\cdots(k+\beta_m)}=\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_m)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}.$$

Partition–HarmonicS

$$\varphi_\infty(f;q)=\prod_{n=1}^{\infty}\left(1-f(n)q^n\right)\Rightarrow\frac{1}{\varphi_\infty(f;q)}=\sum_{\lambda\in\mathcal{P}}q^{|\lambda|}\prod_{\lambda_i\vdash\lambda}f(\lambda_i).$$

Let $f(n)=\frac{t^a}{n^a}$ and $q\to 1$

$$\prod_{n=1}^{\infty}\frac{1}{1-\frac{t^a}{n^a}}=\sum_{k=0}^{\infty}\sum_{l(\lambda)=k}\frac{t^{ak}}{n_\lambda^a}=\sum_{k=0}^{\infty}S_{\mathbf{a}_k}(\infty)t^{ak}.$$

In particular, if $a=m\in\mathbb{N}$ and $m\geq 2$, by considering $\xi_m:=\exp\left(\frac{2\pi\iota}{m}\right)$

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Recurrence of S-sums?

(Thank to Jakob) Blümleim:

$$\begin{aligned}\sum_{\text{perm}} S_{a_1,\ldots,a_k} &= S_{a_1}\cdots S_{a_k} + C_{a_1\wedge a_2,\ldots a_k}\sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3}\cdots S_{a_k}\\ &+C_{a_1\wedge a_2,a_3\wedge a_4,a_5,\ldots a_k}\sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3\wedge a_4}S_{a_5}\cdots S_{a_k}\\ &+\cdots+C_{a_1\wedge a_2\wedge a_3,a_4,\ldots a_k}\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3}S_{a_4}\cdots S_{a_k}\\ &+\cdots+C_{a_1\wedge\cdots\wedge a_k}\sum_{\text{inv term}} S_{a_1\wedge\cdots\wedge a_k},\end{aligned}$$

where

1. "perm" denotes all permuations;
2. "inv perm" denotes all permutations in which a single index in a $\wedge$-contraction is only used once;
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Thus, when $a_1 = \cdots = a_k = a > 0$, we note that

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2. "perm" gives $k!$;
3. "inv perm" gives, for $\pi = \left(\underbrace{\lambda_1, \ldots, \lambda_1}_{l_1}, \ldots, \underbrace{\lambda_r, \ldots \lambda_r}_{l_r}\right) \vdash n$, $\binom{k}{l_1, l_1, \ldots, l_r} \prod_{j=1}^{r} \frac{1}{l_j!}$, where the multinomial comes from all possible permutations while the product is due to uniqueness of single index contraction;
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$$S_{a_k}(N) = \sum_{\pi=\big(\underbrace{\lambda_1,\ldots,\lambda_1}_{l_1},\ldots,\underbrace{\lambda_r,\ldots\lambda_r}_{l_r}\big)\vdash k} \prod_{j=1}^{r} \frac{1}{l_j!}\left(\frac{S_{\lambda_j a}(N)}{\lambda_j}\right)^{l_j}.$$

Γ? By accident and on Wikipedia (containing typos)

$$\Gamma^{(k)}(1) = (-1)^k k! \sum_{\pi=\big(\underbrace{\lambda_1,\ldots,\lambda_1}_{l_1},\ldots,\underbrace{\lambda_r,\ldots\lambda_r}_{l_r}\big)\vdash k} \prod_{j=1}^{r} \frac{\zeta^*(\lambda_i)^{l_j}}{l_j!\lambda_j^{l_j}},\ \zeta^*(a) = \begin{cases} \zeta(a), & \text{if } a \neq 1; \\ \gamma, & \text{if } a = 1. \end{cases}$$

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Recurrence of S-sums?

$$\mathrm{Res}(\Gamma, -n) = \frac{\cdots}{n!}.$$

The poles and residues can be obtained from the formula

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^{\infty} t^{z-1} e^{-t} dt.$$

(This can be seen by expanding exp(-*t*).)

Moreover, the gamma function has the following Laurent expansion at 1

$$\Gamma(z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma^{(n)}(1)}{n!}(z-1)^n,$$

valid for $|z-1| < 1$.

Using the identity

$$\Gamma^{(n)}(1) = (-1)^n n! \sum_{\pi \vdash n} \prod_{i=1}^{r} \frac{\zeta^*(a_i)}{k_i! \cdot a_i} \qquad \zeta^*(x) := \begin{cases} \zeta(x) & x \neq 1 \\ \gamma & x = 1 \end{cases}$$

with partitions

$$\pi = (\underbrace{a_1, \ldots, a_1}_{k_1}, \ldots, \underbrace{a_r, \ldots, a_r}_{k_r}),$$

we have in particular

$$\Gamma(z) = \frac{1}{z} - \gamma + \tfrac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) z - \tfrac{1}{6}\left(\gamma^3 + \frac{\gamma\pi^2}{2} + 2\zeta(3)\right) z^2 + O(z^3).$$

Fourier series expansion [edit]

The logarithm of the gamma function has the following Fourier series expansion

$$\ln\Gamma(x) = \left(\tfrac{1}{2} - x\right)(\gamma + \ln 2) + (1-x)\ln\pi - \tfrac{1}{2}\ln\sin\pi x + \tfrac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n x \cdot \ln n}{n}, \qquad 0 < x < 1,$$

which was for a long time attributed to Ernst Kummer, who derived it in 1847.[3][4] However, it was comparatively recently that it wa

Bell Polynomial

$$\Gamma^{(k)}(1) = Y_k\left(-\gamma, \zeta(2), \ldots, (-1)^k (k-1)!\zeta(k)\right)$$

where, the *complete Bell polynomial* Y_k is defined by

$$Y_k(g_1, \ldots, g_k) = \sum_{(l)\vdash k} \frac{k!}{l_1! \cdots l_k!} \left(\frac{g_1}{1!}\right)^{l_1} \cdots \left(\frac{g_k}{k!}\right)^{l_k},$$

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$$S_{a_k}(N) = \frac{1}{k!} Y_k\left(0!S_a(N), 1!S_{2a}(N), \ldots, (k-1)!S_{ka}(N)\right)$$

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Partial Fractional Decomposition (PFD)

Thank to Jacob:
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$$\prod_{l=1}^{N}\frac{1}{1-a_l z}=\sum_{l=1}^{N}\left(\frac{1}{1-a_l z}\prod_{\substack{j=1\\ j\neq l}}^{N}\frac{1}{1-\frac{a_j}{a_l}}\right),$$

and compares coefficients of z^k of both sides to obtain

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Now, letting $a_l\mapsto\frac{1}{a_l}$, and $a_n:=n^a$

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Special case $a = 1$(Dilcher's case)

$$\prod_{\substack{j=1 \\ j\neq l}}^{N} \frac{j}{j-l} = (-1)^{l-1}\binom{N}{l}.$$

Proof is straightforward.

$$\prod_{\substack{j=1 \\ j\neq l}}^{N} \frac{j}{j-l} = \left(\prod_{\substack{j=1 \\ j\neq l}}^{N} \frac{j}{j-l}\right) \cdot \frac{l}{l}.$$

Question:

$$\prod_{\substack{j=1 \\ j\neq l}}^{N} \frac{j^a}{j^a - l^a}$$

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Special case $a = m \in \mathbb{N}$

Use

$$j^m - l^m = (j - l)(j - \xi_m l) \cdots \left(j - \xi_m^{m-1} l\right)$$

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$$\prod_{\substack{j=1 \\ j \neq l}}^{N} \frac{j^m}{j^m - l^m} = (-1)^{m-1} \prod_{i=0}^{m-1} \frac{N!}{\left(1 - \xi_m^i l\right) \cdots \left((l-1) - \xi_m^i l\right) \xi_m^i l \left((l+1) - \xi_m^i l\right) \cdots \left(N - \xi_m^i l\right)}.$$

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Theorem[Generalization of Dilcher]

If we define

$$\binom{N}{l}_{(m,i)} := \begin{cases} (-1)^{l-1}\binom{N}{l}, & \text{if } \xi_m^i = 1; \\ \frac{N!}{(1-\xi_m^i l)_N} \cdot \frac{1-\xi_m^i}{\xi_m^i}, & \text{if } \xi_m^i \neq 1. \end{cases}$$

Then, for m, k, $N \in \mathbb{N}$

$$S_{\boldsymbol{m}_k}(N) = \sum_{N \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{1}{(i_1 \cdots i_k)^m} = \sum_{l=1}^{N} \binom{N}{l}_{(m,i)} \frac{1}{l^{mk}}.$$

PDF

Kirschenhofer uses

$$\prod_{\substack{j=0\\ j\neq K}}^{N} \frac{1}{1-\frac{t}{j-K}} - 1 = \sum_{m=1}^{\infty} \frac{t^m}{m!} Y_m \left(\ldots, (i-1)! \left(\sum_{\substack{j=0\\ j\neq K}}^{N} \frac{1}{(j-K)^i} \right), \ldots \right).$$

Apparently, it remains hold if letting $K=0$, $t \mapsto t^a$, and $j \mapsto n^a$, namely

$$\prod_{n=1}^{N} \frac{1}{1-\frac{t^a}{n^a}} = \sum_{m=1}^{\infty} \frac{t^{am}}{m!} Y_m \left(\ldots, (i-1)! \sum_{n=1}^{N} \frac{1}{n^{ai}}, \ldots \right) = \sum_{k=1}^{\infty} \frac{t^{ak}}{k!} Y_k \left(\ldots, (i-1)! S_{ai}(N), \ldots \right).$$

Comparing coefficients of t and use PFD leads to:

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Beta function

Blümleim, Klein, Schneider and Stan:

$$B(N,1-t) = \frac{1}{N}\sum_{k=0}^{\infty} t^k S_{1_k}(N).$$

Note from Blümleim

$$B(N,1-t) = \int_0^1 \lambda^{-t}(1-\lambda)^{N-1}\,d\lambda = \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\int_0^1 \log^k\lambda\,(1-\lambda)^{N-1}\,d\lambda,$$

and

$$\int_0^1 \log^k\lambda\,(1-\lambda)^{N-1}\,d\lambda = \frac{(-1)^k k!}{n} S_{1_k}(N).$$

$$\frac{1}{N}\sum_{k=0}^{\infty} t^k S_{1_k}(N) = B(N,1-t) = \frac{\Gamma(N)}{(1-t)\cdots(N-t)}.$$

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$$\prod_{n=1}^{\infty}\frac{1}{\left(1-f\left(n\right)q^{n}\right)}=\sum_{\lambda\in\mathcal{P}}q^{|\lambda|}\prod_{\lambda_i\vdash\lambda}f\left(\lambda_i\right).$$

$$\prod_{n\in X\subseteq\mathbb{N}}\frac{1}{\left(1-f\left(n\right)q^{n}\right)}=\sum_{\lambda\in\mathcal{P}_X}q^{|\lambda|}\prod_{\lambda_i\vdash\lambda}f\left(\lambda_i\right),\ \text{if } f\left(n\right)=0 \text{ when } n\notin X.$$

$$\left.\begin{array}{r}X=[N]=\{1,\ldots,N\}\\ f\left(n\right)=\dfrac{t^a}{n^a}\end{array}\right\}\Rightarrow\prod_{n=1}^{N}\frac{1}{1-\frac{t^a}{n^a}}=\frac{\left(N!\right)^a}{\left(1^a-t^a\right)\cdots\left(N^a-t^a\right)}=\sum_{k=0}^{\infty}S_{a_k}\left(N\right)t^{ak}.$$

Compare:

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Integral Representation

Consider the case $a = 2$

$$B(1+t,1-t) = \Gamma(1+t)\Gamma(1-t) = \sum_{k=0}^{\infty} S_{2_k}(\infty)\, t^{2k}.$$

By integral representation

$$B(1+t,1-t) = \int_0^1 \lambda^{-t}(1-\lambda)^t\, dt = \sum_{k=0}^{\infty} \frac{t^k}{k!}\int_0^1 \log^k\left(\frac{1-\lambda}{\lambda}\right) d\lambda.$$

Therefore,

$$S_{2_k}(\infty) = \frac{1}{(2k)!}\int_0^1 \log^{2k}\left(\frac{1-\lambda}{\lambda}\right) d\lambda.$$

In particular

$$\zeta(2) = \frac{\pi^2}{6} = S_2(\infty) = \frac{1}{2}\int_0^1 \log^2\left(\frac{1-\lambda}{\lambda}\right) d\lambda$$

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Consider the *multiple beta function*, defined by

$$B(\alpha_1, \ldots, \alpha_n) := \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n)},$$

which admits an integral representation, due to the *Dirichlet distribution*

$$B(\alpha_1, \ldots, \alpha_n) = \int_{\Omega_n} \prod_{i=1}^{n} x_i^{\alpha_i - 1} d\mathbf{x},\ \Omega_n := \left\{ (x_1, \ldots, x_K) \in \mathbb{R}_+^n : x_1 + \cdots + x_{n-1} < 1 \text{ and } x_1 + \cdots + x_n = 1 \right\}.$$

Namely,

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Integral Representation

Use

$$B\left(1-\xi_m^0 t, 1-\xi_m^1 t, \ldots, 1-\xi_m^{m-1} t\right) = \frac{\prod_{j=0}^{m-1} \Gamma\left(1-\xi_m^j t\right)}{(m-1)!} = \frac{1}{(m-1)!} \sum_{k=0}^{\infty} S_{\mathbf{m}_k}(\infty) t^{mk}$$

to obtain

$$S_{\mathbf{m}_k}(\infty) = \frac{(-1)^{mk}}{(m-1)!\,(mk)!} \int_0^1 \cdots \int_0^{1-x_1-\cdots-x_{k-2}} \log^{mk}\left(x_1^{\xi_m^0} x_2^{\xi_m^1} \cdots x_{m-1}^{\xi_m^{m-2}} \left(1-x_1-\cdots-x_{m-1}\right)^{\xi_m^{m-1}}\right)$$

and in particular

$$\zeta(m) = \frac{(-1)^m}{(m-1)!m!} \int_0^1 \cdots \int_0^{1-x_1-\cdots-x_{m-2}} \log^m\left(x_1^{\xi_m^0} \cdots x_{m-1}^{\xi_m^{m-2}} \left(1-x_1-\cdots-x_{m-1}\right)^{\xi_m^{m-1}}\right) dx_{m-1} \cdots$$

Integral Representation

Use

$$B\left(1-\xi_m^0 t, 1-\xi_m^1 t, \ldots, 1-\xi_m^{m-1} t\right) = \frac{\prod_{j=0}^{m-1} \Gamma\left(1-\xi_m^j t\right)}{(m-1)!} = \frac{1}{(m-1)!} \sum_{k=0}^{\infty} S_{\mathbf{m}_k}(\infty)\, t^{mk}$$

to obtain

$$S_{\mathbf{m}_k}(\infty) = \frac{(-1)^{mk}}{(m-1)!\,(mk)!} \int_0^1 \cdots \int_0^{1-x_1-\cdots-x_{k-2}} \log^{mk}\left(x_1^{\xi_m^0} x_2^{\xi_m^1} \cdots x_{m-1}^{\xi_m^{m-2}} \left(1-x_1-\cdots-x_{m-1}\right)^{\xi_m^{m-1}}\right)$$

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$$\zeta(m) = \frac{(-1)^m}{(m-1)!m!} \int_0^1 \cdots \int_0^{1-x_1-\cdots-x_{m-2}} \log^m\left(x_1^{\xi_m^0} \cdots x_{m-1}^{\xi_m^{m-2}} \left(1-x_1-\cdots-x_{m-1}\right)^{\xi_m^{m-1}}\right) dx_{m-1} \cdots$$

Extra:Quasi-Shuffle

$$aw_1 * bw_2 = a\left(w_1 * bw_2\right) + b\left(aw_1 * w_2\right) + [a, b]\left(w_1 * w_2\right)$$

$$\sum_{\text{per}\{a_1,\ldots,a_m\}} a_1 \cdots a_m = \sum_{\pi \vdash m} (-1)^{C_\pi} C_\pi \sum_{\text{inv per}\{a_1,\ldots,a_m\}} \pi\left(a_1 \cdots a_m\right),$$

where

$$C_\pi = \prod_{j=1}^{m} \left(\Gamma\left(j\right)\right)^{\pi_j} \text{ for } \pi = \left(\underbrace{1,\ldots,1}_{\pi_1},\ldots,\underbrace{m,\ldots,m}_{\pi_m}\right) \vdash m$$

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What to do?

I need your suggestions!

Thank You!

What to do?

I need your suggestions!

Thank You!

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