Graded Algebras, Algebraic Functions, Planar Trees, and Elliptic Integrals

Vesselin Drensky
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia, Bulgaria
drensky@math.bas.bg

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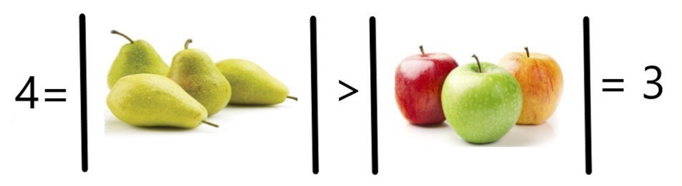
How to measure infinite objects in algebra?

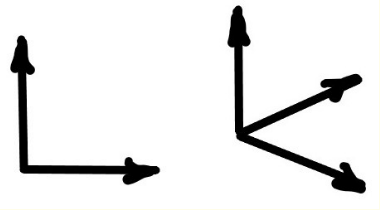
The most natural method to compare finite sets is to use the number of their elements, i.e., the cardinality of the sets.



Very often this method does not work when we compare infinite sets.

Vector spaces

When we measure vector spaces we use their dimension:



$$2 = \dim(\mathbb{R}^2) < \dim(\mathbb{R}^3) = 3.$$

Again, comparing the dimensions may be not successful for vector spaces of infinite dimension.

What to do with infinite dimensional vector spaces?

Why polynomials in three variables are more
than the polynomials in two variables?

First approach

Let K be an arbitrary field and let $K[X_d] = K[x_1, \dots, x_d]$ be the algebra of polynomials in d variables. We shall measure the algebra using the dimensions of the vector spaces of the polynomials of degree $\leq n$, $n = 0, 1, 2, \dots$:

$$g(K[X_d], n) = \binom{n+d}{d} = \frac{(n+d)(n+d-1)\cdots(n+1)}{d!}$$

$$= \frac{n^d}{d!} + \mathcal{O}(n^{d-1}).$$

The function

$$g = g_V : \mathbb{N}_0 \to \mathbb{N}_0$$

is called the growth function of the algebra $K[X_d]$ with respect to the generating vector space $V = KX_d$ with basis $X_d = \{x_1, \dots, x_d\}$.

Disadvantage

The growth function depends on the system of generators. The algebra $K[X_d]$ is generated also by the monomials of first and second degree, i.e., by the vector space $W = V + V^2$. Then

$$g_W(K[X_d], n) = \binom{2n+d}{d} = \frac{2^d n^d}{d!} + \mathcal{O}(n^{d-1}).$$

What is common between both generating functions?

$$\lim_{n\to\infty} \log_n(g_V(K[X_d], n)) = \lim_{n\to\infty} \log_n(g_W(K[X_d], n)) = d.$$

Gelfand-Kirillov dimension

Let R be an algebra generated by the finite dimensional vector space V with basis $\{r_1, \ldots, r_d\}$ and let

$$V_n = V^0 + V^1 + \cdots + V^n = \mathrm{span}\{r_{i_1} \cdots r_{i_m} \mid 0 \leq m \leq n\}.$$

Then the growth function of R with respect to V is

$$g_V(R, n) = \dim(V_n), \quad n = 0, 1, 2, \ldots,$$

and the Gelfand-Kirillov dimension is the upper limit (if it exists)

$$\mathrm{GKdim}(R) = \limsup_{n \to \infty} \log_n(g_V(R, n)).$$

Properties

- R – commutative $\Rightarrow$ $\text{GKdim}(R)$ is an integer equal to the transcendence degree of the algebra R (classics).
- R –associative $\Rightarrow$ $\text{GKdim}(R) \in \{0,1\} \cup [2,\infty]$ and every of these reals is realized as a Gelfand-Kirillov dimension:
  $\text{GKdim}(R) \notin (1,2)$ – Bergman Gap Theorem
  $\text{GKdim}(R) \in [2,\infty)$ is realized:
  W. Borho, H. Kraft, Über die Gelfand-Kirillov Dimension, Math. Ann. 220 (1976), 1-24.

Graded algebras

The polynomial algebra $K[X_d]$ is graded, i.e., every polynomial is a sum of homogeneous polynomials.
The vector space W is graded if it is a direct sum of the form

$$W = W^{(0)} \oplus W^{(1)} \oplus W^{(2)} \oplus \cdots .$$

When the homogeneous components $W^{(n)}$, $n = 0, 1, 2, \ldots$, are finite dimensional, the information for their dimensions is encoded in the formal power series

$$H(W, z) = \sum_{n \geq 0} \dim(W^{(n)}) z^n,$$

called the Hilbert series (or Poincaré series) of W.

The algebra R is graded if

$$R = R^{(0)} \oplus R^{(1)} \oplus R^{(2)} \oplus \cdots ,$$

and

$$R^{(m)} R^{(n)} \subseteq R^{(m+n)}, \quad m, n \geq 0.$$

Usually we assume that $R^{(0)} = K$ or $R^{(0)} = 0$.

Properties of the Hilbert series of commutative algebras

Let R be a finitely generated graded commutative algebra. Then:

- The Hilbert series $H(R,z)$ is a rational function with denominator which is a product of binomials $1-z^m$ (Theorem of Hilbert-Serre).
- If

$$H(R,z) = p(z)\prod \frac{1}{(1-z^{m_i})^{a_i}}, \quad a_i \geq 1, \quad p(z) \in \mathbb{Q}[z],$$

  then the Gelfand-Kirillov dimension $\mathrm{GKdim}(R)$ is equal to the multiplicity of 1 as a pole of $H(R,z)$:

$$\text{If } p(1) \neq 0, \text{ then } \mathrm{GKdim}(R) = \sum a_i.$$

Hilbert series of associative (noncommutative) algebras

$R = K\langle X_d\rangle$ – free associative algebra of rank d (the algebra of polynomials in d noncommutative variables)

$$H(K\langle X_d\rangle, z) = \frac{1}{1-dz} \text{ (rational function);}$$

$$g(K\langle X_d\rangle, n) = 1 + d + d^2 + \cdots + d^n = \frac{1-d^{n+1}}{1-d}$$

(the growth function grows exponentially).

Problem

What can we say about the Hilbert series of a finitely generated graded associative algebra?

Possibilities:

- Rational function;
- Algebraic function (examples??);
- Transcendental function (examples??).

Rational Hilbert series

It is easy to construct finitely generated graded associative algebras with rational Hilbert series.

Theorem of Govorov

Finitely presented monomial algebras have rational Hilbert series.
V.E. Govorov, Graded algebras, Mat. Zametki 12 (1972) 197-204 (Russian), translation in Math. Notes 12(1972) (1973) 552-556.
(Finitely presented monomial algebras are factor algebras of the free associative algebra modulo an ideal generated by a finite number of monomials.)

Approach of Ufnarovskii

Ufnarovskii associates a graph to the set of monomials defining the monomial algebra and estimates the growth function in the language of graph theory.

V.A. Ufnarovskii, Criterion for the growth of graphs and algebras given by words, Mat. Zametki 31 (1982), 465-472 (Russian), translation in Math. Notes 31 (1982), 238-241.

Further generalizations

Recently there are applications to monomial algebras of the theory of regular languages and the theory of finite-state automata which give new results and new proofs of old results providing algebras with rational Hilbert series.

R. La Scala, Monomial right ideals and the Hilbert series of noncommutative modules, J. Symb. Comput. 80 (2017), Part 2, 403-415.

R. La Scala, S.K. Tiwari, Multigraded Hilbert series of noncommutative modules, J. Algebra 516 (2018), 514-544.

Growth of coefficients

What is the possible growth of the coefficients of the Hilbert series?

- Polynomial or exponential growth;
- Can the coefficients have intermediate growth (can they grow faster than polynomially and slower than exponentially)?

It is known that the coefficients of algebraic functions grow polynomially or exponentially. Hence the intermediate growth implies that the Hilbert series is transcendental.

Algebras with intermediate growth – the example of Martha Smith

There exists a two-generated infinite dimensional graded Lie algebra L with Hilbert series

$$H(L,z) = z + \frac{z}{1-z}.$$

The Hilbert series of its universal enveloping algebra $U(L)$ is with intermediate growth of the coefficients:

$$H(U(L),z) = \frac{1}{1-z}\prod_{n\geq 1}\frac{1}{1-z^n}.$$

M. K. Smith, Universal enveloping algebras with subexponential but not polynomially bounded growth, Proc. Amer. Math. Soc. 60 (1976), No. 1, 22–24.

Further results

Lichtman generalizes the result of Martha Smith for different classes of Lie algebras.

A. I. Lichtman, Growth in enveloping algebras, Israel J. Math. 47 (1984), No. 4, 296–304.

Detailed scale to measure the growth

Petrogradsky develops the theory of functions with intermediate growth of the coefficients which are realized as Hilbert series in the known examples of algebras with intermediate growth.

V. M. Petrogradsky, Intermediate growth in Lie algebras and their enveloping algebras, J. Algebra 179 (1996), No. 2, 459–482.

V. M. Petrogradsky, Growth of finitely generated polynilpotent Lie algebras and groups, generalized partitions, and functions analytic in the unit circle, Internat. J. Algebra Comput. 9 (1999), No. 2, 179–212.

Finitely presented algebras of intermediate growth

Every d-generated associative algebra R is a homomorphic image of the free associative algebra $K\langle X_d\rangle$, i.e., there exists an ideal I of $K\langle X_d\rangle$ such that R is a factor algebra of $K\langle X_d\rangle$ modulo the ideal:

$$R \cong K\langle X_d\rangle / I.$$

The algebra is finitely presented if the ideal I is finitely generated.

The algebras in the examples of Martha Smith, Lichtman, and Petrogradsky are not finitely presented.

The example of Ufnarovskii

V. A. Ufnarovskii, Poincaré series of graded algebras (Russian), Mat. Zametki 27 (1980), No. 1, 21–32. Translation. Math. Notes 27 (1980), No. 1, 12-18.

Let $W_1 = \mathrm{Der}(K[x])$ be the Lie algebra of the derivations of the polynomial algebra in one variable over a field K of characteristic 0. This algebra has a graded basis

$$\left\{\delta_{p-1} = x^p\frac{d}{dx} \mid p \geq 0\right\}, \quad \deg\left(x^p\frac{d}{dx}\right) = p-1,$$

and a multiplication

$$[\delta_{p-1}, \delta_{q-1}] = \delta_{p-1}\delta_{q-1} - \delta_{q-1}\delta_{p-1} = \left[x^p\frac{d}{dx}, x^q\frac{d}{dx}\right]$$

$$= (q-p)x^{p+q-1}\frac{d}{dx} = (q-p)\delta_{p+q-2}.$$

Let L be the Lie subalgebra of W_1 generated by δ_1 and δ_2. It has a basis

$$\{\delta_p \mid p = 1, 2, \ldots\},$$

and the derivations δ_p may be defined inductively by

$$\delta_{p+1} = \frac{1}{p-1}[\delta_1, \delta_p], \quad p = 2, 3, \ldots.$$

It has turned out that in this notation the algebra L has defining relations

$$[\delta_2, \delta_3] = \delta_5 \text{ and } [\delta_2, \delta_5] = 3\delta_7.$$

The universal enveloping algebra $U(L)$ of L is an associative algebra with basis

$$\{f_1^{n_1}\cdots f_p^{n_p} \mid n_i \geq 0\},$$

generated by $f_1 = x$ and $f_2 = y$, where

$$f_{p+1} = \frac{1}{p-1}(f_1 f_p - f_p f_1), \quad p = 2, 3, \ldots .$$

The algebra $U(L)$ is a factor algebra of the free algebra $K\langle x, y\rangle$ modulo the ideal generated by

$$(f_2 f_3 - f_3 f_2) - f_5 \text{ and } (f_2 f_5 - f_5 f_2) - 3f_7.$$

If we assume that $\deg f_p = p$, then the Hilbert series of $U(L)$ is

$$H(U(L), z) = \prod_{n\geq 1} \frac{1}{1-z^n} = \sum_{n\geq 0} \mathcal{P}_n z^n.$$

The positive integer $\mathcal{P}_n$ is equal to the number of partitions of the integer n

$$\lambda = (\lambda_1, \ldots, \lambda_m), \quad \lambda_1 + \cdots + \lambda_m = n, \quad \lambda_1 \geq \cdots \geq \lambda_m \geq 0.$$

By the classical formula of Hardy-Ramanudjan in 1918 (established independently by Uspensky in 1920)

$$\mathcal{P}_n \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2}{3}n}\right)$$

and $\mathcal{P}_n$ is of intermediate growth.

G. H. Hardy, S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. Lond. Math. Soc. (2) 17 (1918), 75-115.

J. V. Uspensky, Asymptotic formulae for numerical functions which occur in the theory of partitions (Russian), Bull. Acad. Sci. URSS 14 (1920), 199-218.

Examples of algebraic Hilbert series

Let R_1 be the subalgebra of the free associative algebra $K\langle X_2\rangle$ consisting of the polynomials $f(x_1, x_2)$ with the property

$$f(x_1, x_1 + x_2) = f(x_1, x_2),$$

and let R_2 be the subalgebra of R_1 of all $f(x_1, x_2)$ such that

$$f(x_1 + x_2, x_2) = f(x_1, x_2).$$

Then:

$$H(R_2, z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}.$$

G. Almkvist, W. Dicks, E. Formanek, Hilbert series of fixed free algebras and noncommutative classical invariant theory, J. Algebra 93 (1985), 189-214.

$$H(R_1, z) = \frac{1 - \sqrt{1 - 4z^2}}{z(2z - 1 + \sqrt{1 - 4z^2})}.$$

V. Drensky, C.K. Gupta, Constants of Weitzenböck derivations and invariants of unipotent transformations acting on relatively free algebras, J. Algebra 292 (2005), 393-428.

Relation with noncommutative invariant theory

The algebra R_2 is an algebra of invariants:
$R_2 = K\langle X_2\rangle^{SL_2(K)}$ is the algebra of invariants in $K\langle X_2\rangle$ of the special linear group $SL_2(K)$, acting canonically on the two-dimensional vector space KX_2.
The algebra R_1 is also an algebra of invariants: $R_1 = K\langle X_2\rangle^{UT_2(K)}$, where

$$UT_2(K) = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in K \right\}.$$

Remark

The algebras R_1 and R_2 are not finitely generated but they are free associative algebras.

D.R. Lane, Free Algebras of Rank Two and Their Automorphisms, Ph.D. Thesis, Bedford College, London, 1976.

V.K. Kharchenko, Algebras of invariants of free algebras (Russian), Algebra Logika 17 (1978), 478-487. Translation. Algebra Logic 17 (1978), 316-321.

Generating sets of the algebras R_1 and R_2

The algebra R_2 has a generating set which is built inductively.
(1) We start with $[x_1, x_2] = x_1x_2 - x_2x_1$.
(2) If $w_1, \ldots, w_m$ are already constructed elements of the generating set (allowing repetitions among these elements), we add to the system of generators the element

$$x_1w_1 \cdots w_mx_2 - x_2w_1 \cdots w_mx_1$$

and continue further.
The algebra R_1 is generated by the generating system of R_2 and the element x_1.

Problem

Up till now we discussed Hilbert series of graded homomorphic images of polynomial algebras and free associative algebras. What will happen if we consider free algebras in other classes? One of the most important algebras from this point of view are relatively free algebras of varieties of associative or nonassociative algebras. We shall restrict our considerations to varieties of associative algebras over a field K of characteristic 0.

Polynomial identities and varieties of algebras

The polynomial $f(x_1, \ldots, x_d) \in K\langle X\rangle$, $X = \{x_1, x_2, \ldots\}$, is a polynomial identity for the associative algebra R if

$$f(r_1, \ldots, r_d) = 0 \text{ for all } r_1, \ldots, r_d \in R.$$

The set $I(R)$ of all polynomial identities of R is a T-ideal of $K\langle X\rangle$, (i.e., an ideal which is invariant under all endomorphisms of $K\langle X\rangle$).

The factor algebra

$$F(\mathrm{var}(R)) = F(\mathfrak{R}) = K\langle X\rangle/I(R)$$

is the relatively free algebra (of countable rank) in the variety of algebras $\mathfrak{R} = \mathrm{var}(R)$ generated by the algebra R. The d-generated subalgebra

$$F_d(\mathrm{var}(R)) = F_d(\mathfrak{R}) = K\langle X_d\rangle/(K\langle X_d\rangle \cap I(R))$$

is the relatively free algebra of rank d in $\mathfrak{R}$.

Properties of $F_d(\mathfrak{R})$

- If A is a d-generated algebra in $\mathfrak{R}$ (i.e., A satisfies all polynomial identities of R), then A is a homomorphic image of $F_d(\mathfrak{R})$.
- $\text{GKdim}(F_d(\mathfrak{R}))$ is a nonnegative integer
  A. Berele, Homogeneous polynomial identities, Israel J. Math. 42 (1982), 258-272. ($\text{GKdim}(F_d(\mathfrak{R})) < \infty$)
  V.T. Markov, The Gelfand-Kirillov dimension: nilpotency, representability, non-matrix varieties (Russian), Siberian School on Varieties of Algebraic Systems, Abstracts, Barnaul, 1988, 43-45. Zbl. 685.00002. ($\text{GKdim}(F_d(\mathfrak{R}))$ is an integer)

- The algebra $F_d(\mathfrak{R})$ is graded and its Hilbert series is a rational function.
  A.Ya. Belov, Rationality of Hilbert series of relatively free algebras (Russian), Uspekhi Mat. Nauk 52 (1997), No. 2, 153-154. Translation: Russian Math. Surveys 52 (1997), 394-395.
- The Hilbert series of $F_d(\mathfrak{R})$ is of the same kind as the Hilbert series of commutative graded algebras.
  A. Berele, Applications of Belov's theorem to the cocharacter sequence of p.i. algebras, J. Algebra 298 (2006), 208-214.

PI-algebras

The class of PI-algebras (algebras with polynomial identities) is quite large. (It contains all commutative and all finite dimensional algebras.) Nevertheless it shares a lot of the properties of commutative and finite dimensional algebras. In particular, it has nice structural and combinatorial theory.

Example: Theorem of Berele

The Gelfand-Kirillov dimension of a finitely generated PI-algebra R is finite.

Invariant theory for relatively free algebras

The general linear group $GL_d(K)$ acts on the vector space KX_d and this action is extended diagonally on $F_d(\mathfrak{R})$:

$$g(f(x_1,\dots,x_d)) = f(g(x_1),\dots,g(x_d)), \quad f \in F_d(\mathfrak{R}), g \in GL_d(K).$$

If G is a subgroup of $GL_d(K)$, then

$$F_d^G(\mathfrak{R}) = \{f \in F_d(\mathfrak{R}) \mid g(f) = f \text{ for all } g \in G\}$$

is the subalgebra of G-invariants in $F_d(\mathfrak{R})$.

Differences with commutative invariant theory

For fixed $\mathfrak{R}$, the condition that $F_d^G(\mathfrak{R})$ is finitely generated for important classes of groups is a very strong restriction on $\mathfrak{R}$.

G – any finite group

The case when $F_d^G(\mathfrak{R})$, $d > 1$, is finitely generated for every finite group is described in a series of papers by Kharchenko (1984), L'vov (1969), Anan'in (1977), Tonov (1981), Drensky (1993), and Fisher-Susan Montgomery (1986).

G – reductive

The algebra $F_d^G(\mathfrak{R})$, $d > 1$, is finitely generated for every reductive group G if and only if $\mathfrak{R}$ satisfies the identity of Lie nilpotency.
M. Domokos, V. Drensky, A Hilbert-Nagata theorem in noncommutative invariant theory, Trans. Amer. Math. Soc. 350 (1998), 2797-2811.

Common properties with commutative invariant theory

If G is a subgroup of $GL_m(K)$ such that for all d and all rational actions of $GL_m(K)$ on KX_d, the algebra of invariants $K[X_d]^G$ is finitely generated, then for all such actions of $GL_d(K)$ the Hilbert series $H(F_d^G(\mathfrak{R}), z)$ is a rational function of the kind which appears in the case of commutative algebra.

M. Domokos, V. Drensky, Rationality of Hilbert series in noncommutative invariant theory, International J. Algebra and Computations 27 (2017), No. 7, 831-848.

Another type of examples

W. Borho, H. Kraft, Über die Gelfand-Kirillov Dimension, Math. Ann. 220 (1976), 1-24.
V. Drensky, Free Algebras and PI-Algebras, Springer-Verlag, Singapore, 2000, Section 9.4.
Let $J \subset \mathbb{N}_0$ and let R be the algebra generated by x, y with basis

$$\{x^m, x^m y x^n, x^m y x^j y x^n \mid m, n \geq 0, j \in J\}$$

and such that all other monomials are equal to 0.

The Hilbert series of R is

$$H(R,z)=\frac{1}{1-z}+\frac{2z}{(1-z)^2}+\frac{z^2}{(1-z)^2}h(z),\quad h(z)=\sum_{j\in J}z^j.$$

- For suitable choices of the set J the function $h(z)$ becomes transcendental, and hence the same holds for the Hilbert series $H(R,z)$.
- For any real $\alpha\in[2,3]$ there exists a set J such that $\mathrm{GKdim}(R)=\alpha$.
- The algebra R is PI. It satisfies

$$[x_1,x_2][x_3,x_4][x_5,x_6]=0.$$

Generalization of the construction

Theorem. Let

$$f(z)=\sum_{n\geq 0}a_nz^n\in\mathbb{Z}[[z]]$$

be a formal power series with nonnegative integer coefficients such that there exists a positive integer d with the property $a_n\leq d^n$. Then there exists a $(d+1)$-generated graded algebra R, such that

$$H(R,z)=\frac{1}{1-dz}+\frac{z}{(1-dz)^2}+\frac{z^2f(z)}{(1-z)^{dp}(1-dz)^q},$$

$p,q\geq 0,\ p+q\leq 2.$

Theorem. If in the notation of the previous theorem there exists a positive integer d, such that $a_n \leq \binom{n+d-1}{d-1}$, then there exists a $(d+1)$-generated graded algebra R with

$$H(R,z) = \frac{1}{(1-z)^d} + \frac{z}{(1-z)^{2d}} + \frac{z^2 f(z)}{(1-z)^{dp}}, \quad 0 \leq p \leq 2.$$

Question (Roberto La Scala)

How to construct graded algebras with Hilbert series which are algebraic but are not rational functions?

Answer

We can apply the previous two theorems. For this purpose we need algebraic but not rational power series with nonnegative integer coefficients.

Warning

The condition that a power series with nonnegative integer coefficients is algebraic is very restrictive.

Theorem (Fatou, 1906)

If the coefficients of a power series are nonnegative integers and are bounded polynomially, then the series is either rational or transcendental.

P. Fatou, Séries trigonométriques et séries de Taylor, Acta Math. 30 (1906), 335-400.

Considérons une série de TAYLOR à coefficients entiers; je dis qu'*elle ne peut représenter une fonction algébrique que si son rayon de convergence est plus petit que l'unité, à moins qu'elle ne soit égale à une fraction rationnelle dont tous les pôles sont des racines de l'unité.*

JFM 37.0283.01

Fatou, P.
Séries trigonométriques et séries de *Taylor*. (French)
Acta Math. 30, 335-400.

Published: **(1906)**

Eine *Taylor*sche Reihe mit ganzen Koeffizienten kann nur dann eine algebraische Funktion darstellen, wenn ihr Konvergenzradius kleiner als Eins ist, wofern sie nicht gleich einem rationalen Bruche ist, dessen Pole sämtlich Wurzeln der Einheit sind. –

Weltzien, Prof. (Zehlendorf)

Corollary

If R is a finitely generated graded algebra of finite Gelfand-Kirillov dimension, then its Hilbert series is either rational or transcendental.

Theorem of Berele

Finitely generated PI-algebras have a finite Gelfand-Kirillov dimension.

A. Berele, Homogeneous polynomial identities, Israel J. Math. 42 (1982), 258-272.

Corollary

If R is finitely generated graded PI-algebra, then its Hilbert series is either rational or transcendental function.

Problem

How to construct algebraic power series which are with nonnegative integer coefficients and are not rational functions?

Idea

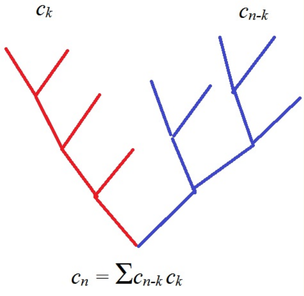
We start with a sequence of finite sets of objects A_n, $n = 0, 1, 2, \ldots$, for which we know (or can prove), that the generating fucntion

$$f(z) = \sum_{n \geq 0} |A_n| z^n$$

of the sequence $|A_n|$, $n = 0, 1, 2, \ldots$, is algebraic but not rational.

Example – Catalan numbers

The n-th Catalan number is equal to the number of planar binary rooted trees with n leaves.



Therefore,

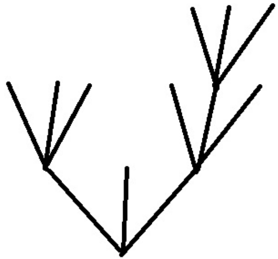
$$c(z) = \sum_{n\geq 1} c_n z^n, \quad c^2(z) = \sum_{n\geq 2}\sum_{k=1}^{n-1} c_k c_{n-k} z^n = c(z) - z,$$

$$c^2(z) - c(z) + z = 0, \quad c(z) = \frac{1-\sqrt{1-4z}}{2},$$

$$c_n = \frac{1}{n}\binom{2n-2}{n-1}, \quad n = 1, 2, \ldots.$$

Similar example – k-ary trees, $k > 2$

These are planar rooted trees such that every vertex (which is not a leave) is an origin of exactly k branches.



$k = 3$

The generating function $f(z)$ of k-ary trees is the solution of the equation

$$f^k(z) - f(z) + z = 0,$$

satisfying the condition $f(0) = 0$.

One more example – arbitrary trees

We consider trees such that every vertex (which is not a leave) is an origin of at least 2 branches. The generating function of such trees satisfies the equation

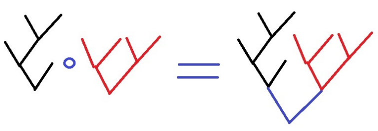
$$\frac{f^2(z)}{1-f(z)} - f(z) + z = 0,$$

$$f(z) = \frac{1+z-\sqrt{1-6z+z^2}}{4}$$

and this is the generating function of the super-Catalan numbers. See the sequence A001003 in The On-Line Encyclopedia of Integer Sequences – https://oeis.org/.

Trnaslation in the language of algebra

We may turn the set of planar binary rooted trees in a nonassociative groupoid (or nonassociative magma):



$$\Downarrow$$

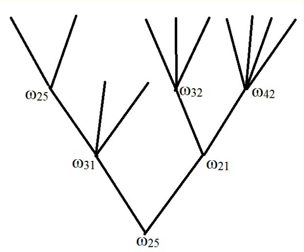
$$((x(xx))x)\circ((xx)(xx))=((x(xx))x)((xx)(xx))$$

Ω-magmas

For any $n \geq 2$ we fix a finite set of n-ary operations

$$\Omega_n = \{\omega_{n1}, \dots, \omega_{np_n}\}, \quad \Omega = \bigcup_{n \geq 2} \Omega_n.$$

As in the case of binary trees, we may consider Ω-trees. The difference is that the vertices which are not leaves are labeled by the elements of Ω,taking into account the n-arity of the operations. Then we can give the set of Ω-trees the structure of an Ω-magma.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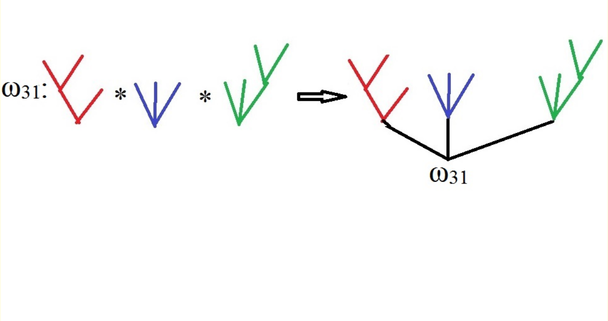
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Results of Vesselin Drensky and Chavdar Lalov: project in the frames of the High School Student Institute of Mathematics and Informatics

Let $p(z) = p_2 z^2 + p_3 z^3 + \cdots$ be the generating function of the set Ω. Then the generating function of the Ω-trees (which counts the trees with a given number of leaves) satisfies the equation

$$p(f(z)) - f(z) + z = 0.$$

Therefore, if $p(z)$ is algebraic, under minimal restrictions on $p(z)$, the function $f(z)$ will be also algebraic.

Theorem of Kurosh (1947, 1969)

Every submagma of a free Ω-magma is also free.

A.G. Kurosh, Non-associative free algebras and free products of algebras (Russian) Rec. Math. (Mat. Sbornik) N.S. 20(62) (1947), 239-262.

A.G. Kurosh, Multiple operator rings and algebras (Russian), Uspehi Mat. Nauk (Russian Math. Surveys) 24 (1969), No. 1(145), 3-15.

Theorem

The set of all Ω-trees is a free Ω-magma, and the set of all Ω-trees with number of leaves divisible by s is a submagma. If $f^{(s)}(z)$ is the generating function of this set, then the generating function $g^{(s)}(z)$ of the free generating set satisfies the equation

$$p(f^{(s)}(z)) - f^{(s)}(z) + g^{(s)}(z) = 0.$$

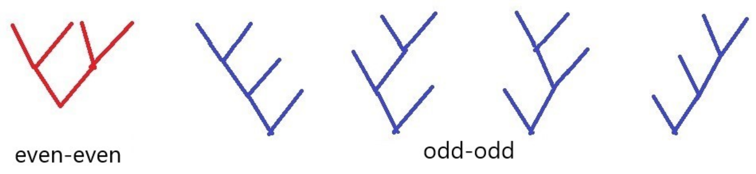
Jointly with Chavdar Lalov, a secondary school student at the Mathematical School "Geo Milev", Pleven, we found a method, starting with an algebraic generating function $p(z)$ of the set of operations Ω (given explicitly or with its equation) to find the generating function $g^{(s)}(z)$ or its equation. Again, under natural restrictions on $p(z)$ it has turned out that $g^{(s)}(z)$ is algebraic but not rational.

The starting point of the project with Chavdar Lalov

Planar binary trees with even number of leaves.

We consider the set E of all planar binary trees with even number of leaves. We may identify E with the set of all words of even length in the free magma $M(x)$ generated by x. There are two possibilities for the number of leaves in the two branches of the tree $(u)(v) \in E$:

$$\text{(even-even)}$$
$$E_0 = \{(u)(v) \mid |u| \equiv |v| \equiv 0 \pmod 2\}$$
$$\text{(odd-odd)}$$
$$E_1 = \{(u)(v) \mid |u| \equiv |v| \equiv 1 \pmod 2\}.$$



Problem. We consider the planar binary rooted trees with even number $2n$ of leaves. Which are more – the trees of type (even-even) or of type (odd-odd)?

Solution

The set E is a submagma of the free magma $M(x)$ and therefore (by the theorem of Kurosh) is free. Its generating set consists of the (odd–odd) trees. If $e(z)$ and $g(z)$ are, respectively, the generating functions of E and of the generators of E, then the generating function of E is

$$e(z) = \sum_{n\geq 1} c_{2n}z^{2n} = \frac{1}{2}(c(z) + c(-z)) = c(g(z)).$$

We consider $g(z)$ as an unknown, solve the equation, and obtain

$$g(z) = \frac{1}{4}c(4z^2), \quad g_{2n} = \frac{1}{4^{n-1}}c_n.$$

Applying the Stirling formula for $n!$ after some calculations we obtain

$$\frac{g_{2n}}{c_{2n}} \approx \frac{1}{2}\sqrt{\frac{2n-1}{n-1}}, \quad \lim_{n\to\infty} \frac{g_{2n}}{c_{2n}} = \frac{\sqrt{2}}{2} \approx 0.707105.$$

Hence the trees of type (odd-odd) are much more than the trees of type (even-even).

V. Drensky, R. Holtkamp, Planar trees, free nonassociative algebras, invariants, and elliptic integrals, Algebra and Discrete Mathematics (2008), No. 2, 1-41.

Another approach

Roberto La Scala, Dmitri Piontkovski, Sharwan K. Tiwari, Noncommutative algebras, context-free grammars and algebraic Hilbert series, Journal of Symbolic Computation (to appear).

The authors produce noncommutative finitely generated monomial algebras whose Hilbert series are algebraic functions. The approach is based on the concept of graded homology and the theory of unambiguous context-free grammars. Explicit examples are provided.

Elliptic integrals

We consider the absolutely free nonassociative algebra $K\{X_2\}$ (in which $uv \neq vu$ and $(uv)w \neq u(vw)$). As in the case of the free associative algebra $K\langle X_2\rangle$, which we have already considered, let R_1 and R_2 be the subalgebras of $K\{X_2\}$ defined by

$$R_1 = \{f(x_1,x_2) \in K\{X_2\} \mid f(x_1,x_1+x_2) = f(x_1,x_2)\},$$

$$R_2 = \{f(x_1,x_2) \in R_1 \mid f(x_1+x_2,x_2) = f(x_1,x_2)\}.$$

Remark

As in the case of the free associative algebra, the algebras R_1 and R_2 are not finitely generated.

Theorem

The Hilbert series of the algebras R_1 and R_2 are elliptic integrals:

$$H(R_1, z) = \int_0^1 \cos^2(\pi u) \left(1 - \sqrt{1 - 8z\cos(2\pi u)}\right) du,$$

$$H(R_2, z) = \int_0^1 \sin^2(2\pi u) \left(1 - \sqrt{1 - 8z\sin(2\pi u)}\right) du$$

The proofs use noncommutative analogue of the Molien-Weyl integral formula for the Hilbert series in classical invariant theory. V. Drensky, R. Holtkamp, Planar trees, free nonassociative algebras, invariants, and elliptic integrals, Algebra and Discrete Mathematics (2008), No. 2, 1-41.

Gradings of free associative algebras

(Byproduct of a project on algebras with polynomial identity born in discussions with Olga Finogenova and Mikhail Zaicev)
Let

$$g(X,z)=\sum_{m\geq 1} d_m z^m,\quad d_m\in\mathbb{N}_0,$$

be a power series with nonnegative integer coefficients. We assume that the free associative algebra $K\langle X\rangle$ is generated by the set

$$X=\bigcup_{m\geq 1}\{x_{m1},\dots,x_{md_m}\}$$

and $\deg(x_{mi})=m$.

Theorem

The Hilbert series of $K\langle X\rangle$ is

$$H(K\langle X\rangle, z) = \sum_{n\geq 0} b_n z^n = \frac{1}{1-g(X,z)}.$$

Problems

- Express explicitly the coefficients b_n in terms of d_n.
- Find the asymptotics of the coefficients b_n.
- Does $\lim_{n\to\infty} \sqrt[n]{b_n}$ exist? Is it rational/algebraic/transcendental?

Theorem

Let the series

$$g(X,z)=\sum_{m\geq 1} d_m z^m,\quad d_m\in\mathbb{N}_0,$$

have a nonzero radius of convergence and let the integers d_m are coprime. Then for the coefficients b_n of the Hilbert series of $K\langle X\rangle$ we have that

$$\lim_{n\to\infty}\sqrt[n]{b_n}$$

exists and is equal to $1/\alpha$, where α is the positive solution of the equation

$$g(X,z)=\sum_{m\geq 1} d_m z^m=1.$$

Transcendental numbers and lacunary series

(lacuna (Latin) = gap)

Liouville number is a real number α with the property that, for every positive integer n, there exist infinitely many pairs of integers (p, q) with $q > 1$ such that

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Liouville showed that such numbers are transcendental, thus establishing the existence of transcendental numbers for the first time.

J. Liouville, Sur des classes très étendues de quantités dont valeur n'est ni algébrique, ni même réducible à des irrationelles algébriques, C.R. Acad. Sci., Paris, Sér. A 18 (1844), 883-885. J. Math. Pures Appl. 16 (1851), 133-142.

Example

The Liouville constant $\alpha = \sum_{k \geq 1} \frac{1}{10^{k!}}$ is transcendental.

Generalizations

Theorem. (A partial case of a result of Cohn) Let

$$f(z) = \sum_{i \geq 0} d_i z^{k_i}, \quad 0 \neq d_i \in \mathbb{N}_0, \quad 0 \leq k_0 < k_1 < \cdots ,$$

$$\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = \lim_{i \to \infty} \frac{k_{i+1}}{\log(\max\{d_0, \dots, d_i\})} = \infty.$$

Then $f(\alpha)$ is transcendental for every nonzero algebraic α within the circle of convergence of $f(t)$.

H. Cohn, Note on almost-algebraic numbers, Bull. Am. Math. Soc. 52 (1946), 1042-1045.

For more recent results see, e.g., the references in

H. Kaneko, Algebraic independence of the values of power series with unbounded coefficients, Ark. Mat. 55 (2017), No. 1, 61-87.

Corollary

If the series

$$g(X,z)=\sum_{m\geq 1} d_m z^m$$

with d_m coprime satisfies the conditions of the theorem of Cohn, then the limit $\lim_{n\to\infty}\sqrt[n]{b_n}$ for the coefficients b_n of the Hilbert series of $K\langle X\rangle$ is a transcendental number.

Problem

(The most important case in the original project on algebras with polynomial identity)
Is the limit $\lim_{n\to\infty}\sqrt[n]{b_n}$ transcendental for the series

$$g(X,z)=\sum_{m\geq 1} z^{m^2}?$$